

SOLUTIONS OF FULLY NONLINEAR NONLOCAL SYSTEMS

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ABSTRACT. In this paper we consider the system involving fully nonlinear nonlocal operators:

$$\begin{cases} F_\alpha(u(x)) = C_{n,\alpha} PV \int_{R^n} \frac{G(u(x)-u(y))}{|x-y|^{n+\alpha}} dy = f(v(x)), \\ F_\beta(v(x)) = C_{n,\beta} PV \int_{R^n} \frac{G(v(x)-v(y))}{|x-y|^{n+\beta}} dy = g(u(x)). \end{cases}$$

A *narrow region principle* and a *decay at infinity* for the system for carrying on the method of moving planes are established. Then we prove the radial symmetry and monotonicity for positive solutions to the nonlinear system in the whole space. Non-existence of positive solutions to the nonlinear system on a half space is proved.

1. INTRODUCTION

In this paper, we consider the nonlinear system involving fully nonlinear nonlocal operators:

$$\begin{cases} F_\alpha(u(x)) = f(v(x)), \\ F_\beta(v(x)) = g(u(x)), \end{cases}$$

with

$$F_\alpha(u(x)) = C_{n,\alpha} PV \int_{R^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy,$$

where PV stands for the Cauchy principal value, G is at least local Lipschitz continuous, $G(0) = 0$ and $0 < \alpha, \beta < 2$. The operators F_α was introduced by Caffarelli and Silvestre in [CS1].

In order the integral to make sense, we require

$$u \in C_{loc}^{1,1} \cap L_\alpha, \quad v \in C_{loc}^{1,1} \cap L_\beta$$

with

$$L_\alpha = \{u : R^n \rightarrow R \mid \int_{R^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty\},$$

and L_β having a similar meaning.

In the special case when $G(\cdot)$ is an identity map, F_α becomes the usual fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. The nonlocal nature of fractional operators makes them difficult to study. To circumvent this, Caffarelli and Silvestre [CS] introduced the *extension method* which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions. This method has been applied successfully to treat equations involving

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the fractional Laplacian and a series of fruitful results has been obtained (see [BCPS], [CZ], etc.). One can also use *the integral equations method*, such as *the method of moving planes in integral forms* (see [CC], [CD], [ZCCY], [LZ], [LZr]) and *regularity lifting* to investigate equations involving fractional Laplacian by showing that they are equivalent to corresponding integral equations (see [CFY], [CLO], [CLO1] and the references therein). For more articles concerning the method of moving planes for nonlocal equations and for integral equations, see [FL], [HLZ], [HWY], [LLM], [LZ2], [MC], [MZ], [LZ] and the references therein.

For the fully nonlinear nonlocal equations, so far as we know, there is neither any corresponding *extension method* nor equivalent integral equations that one can work at. A probable reason is that very few results were obtained for fully nonlinear nonlocal operator. In [CLL], Chen, Li and Li developed a new method that can deal directly with these nonlocal operators. Inspired by the idea, we extend the method in [CLL] to fully nonlinear nonlocal systems and consider the nonlinear systems involving fully nonlinear nonlocal operators

$$(1.1) \quad \begin{cases} F_\alpha(u(x)) = f(v(x)), \\ F_\beta(v(x)) = g(u(x)), \\ u(x) > 0, v(x) > 0, \end{cases} \quad x \in R^n,$$

and

$$(1.2) \quad \begin{cases} F_\alpha(u(x)) = f(v(x)), \\ F_\beta(v(x)) = g(u(x)), \\ u(x) \equiv 0, v(x) \equiv 0, \end{cases} \quad \begin{matrix} x \in R_+^n, \\ x \notin R_+^n, \end{matrix}$$

where f and g are nonnegative continuous and nondecreasing functions.

We first establish the *narrow region principle* and *decay at infinity* for the system which play important roles in carrying out the method of moving planes. To state them, denote by

$$T_\lambda = \{x \in R^n | x_1 = \lambda\}$$

the moving plane,

$$\Sigma_\lambda = \{x \in R^n | x_1 < \lambda\}$$

the left region of the plane T_λ ,

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

the reflection of x about T_λ , and denote

$$U_\lambda(x) = u_\lambda(x) - u(x) \text{ and } V_\lambda(x) = v_\lambda(x) - v(x).$$

For simplicity of notations, we stand for $U_\lambda(x)$ by $U(x)$ and $V_\lambda(x)$ by $V(x)$ in the sequel.

Theorem 1.1. (*Narrow Region Principle*) *Let Ω be a bounded narrow region in Σ_λ contained in*

$$\{x | \lambda - l < x_1 < \lambda\}$$

with small $l > 0$. Suppose that $U(x) \in L_\alpha \cap C_{loc}^{1,1}(\Omega)$, $V(x) \in L_\beta \cap C_{loc}^{1,1}(\Omega)$, and $U(x), V(x)$ are lower semi-continuous on $\bar{\Omega}$. If $c_i(x) \leq 0, i = 1, 2$, are bounded from below in Ω , $U(x)$ and $V(x)$ satisfy

$$(1.3) \quad \begin{cases} F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + c_1(x)V(x) \geq 0, \\ F_\beta(v_\lambda(x)) - F_\beta(v(x)) + c_2(x)U(x) \geq 0, \\ U(x), V(x) \geq 0, \\ U(x^\lambda) = -U(x), V(x^\lambda) = -V(x), \end{cases} \quad \begin{array}{l} x \in \Omega, \\ x \in \Sigma_\lambda \setminus \Omega, \\ x \in \Sigma_\lambda, \end{array}$$

then we have for sufficiently small l ,

$$(1.4) \quad U(x), V(x) \geq 0 \text{ in } \Omega;$$

if Ω is unbounded, the conclusion still holds under the conditions

$$\liminf_{|x| \rightarrow \infty} U(x), \liminf_{|x| \rightarrow \infty} V(x) \geq 0;$$

furthermore, if $U(x)$ or $V(x)$ attains 0 somewhere in Σ_λ , then

$$(1.5) \quad U(x) = V(x) \equiv 0, \quad x \in R^n.$$

We call (1.5) the strong maximum principle later. As we can see from the proof, to ensure (1.5), Ω does not need to be narrow.

Theorem 1.2. (*Decay at Infinity*) Let Ω be a bounded or unbounded domain in R^n . Assume that $U(x) \in C_{loc}^{1,1}(\Omega) \cap L_\alpha(R^n)$, $V(x) \in C_{loc}^{1,1}(\Omega) \cap L_\beta(R^n)$, $U(x)$ and $V(x)$ are lower semi-continuous on $\bar{\Omega}$. If $U(x)$ and $V(x)$ satisfy

$$(1.6) \quad \begin{cases} F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + c_1(x)V(x) \geq 0, \\ F_\beta(v_\lambda(x)) - F_\beta(v(x)) + c_2(x)U(x) \geq 0, \\ U(x), V(x) \geq 0, \\ U(x^\lambda) = -U(x), \\ V(x^\lambda) = -V(x), \end{cases} \quad \begin{array}{l} x \in \Omega, \\ x \in \Sigma_\lambda \setminus \Omega, \\ x \in \Sigma_\lambda, \end{array}$$

with

$$(1.7) \quad c_1(x) \sim o\left(\frac{1}{|x|^\alpha}\right), \quad c_2(x) \sim o\left(\frac{1}{|x|^\beta}\right), \text{ for } |x| \text{ large,}$$

and

$$c_i(x) \leq 0, \quad i = 1, 2,$$

then there exists a constant $R_0 > 0$ depending only on $c_i(x)$ such that if

$$U(\tilde{x}) = \min_{\Omega} U(x) < 0 \text{ and } V(\bar{x}) = \min_{\Omega} V(x) < 0,$$

then

$$(1.8) \quad |\tilde{x}| \leq R_0 \text{ or } |\bar{x}| \leq R_0.$$

Based on Theorems 1.1 and 1.2, we apply the *method of moving planes* to obtain symmetry and monotonicity of positive solutions to (1.1) in R^n , as well as nonexistence of positive solutions to (1.2) on the half space.

Theorem 1.3. Assume that $u(x) \in L_\alpha(R^n) \cap C_{loc}^{1,1}(R^n)$ and $v(x) \in L_\beta(R^n) \cap C_{loc}^{1,1}(R^n)$ are positive solutions of system (1.1). Suppose that for some $\gamma, \tau > 0$,

$$(1.9) \quad v(x) = o\left(\frac{1}{|x|^\gamma}\right), \quad u(x) = o\left(\frac{1}{|x|^\tau}\right), \quad \text{as } |x| \rightarrow \infty,$$

and

$$(1.10) \quad f'(s) \leq s^q, \quad g'(t) \leq t^p, \quad \text{with } q\gamma \geq \alpha, \quad p\tau \geq \beta.$$

Then $u(x)$ and $v(x)$ must be radially symmetric and monotone decreasing about some point x_0 in R^n .

Theorem 1.4. Assume that $u(x) \in L_\alpha \cap C_{loc}^{1,1}(R_+^n)$, $v(x) \in L_\beta \cap C_{loc}^{1,1}(R_+^n)$ are nonnegative solutions of system (1.2). Suppose

$$(1.11) \quad \lim_{|x| \rightarrow \infty} u(x) = 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 0,$$

$f(v), g(u)$ are Lipschitz continuous in the range of $v(x), u(x)$ respectively, and $f(0) = 0, g(0) = 0$. Then $u(x) \equiv 0, v(x) \equiv 0$.

In section 2, we prove Theorems 1.1 and 1.2 with a key ingredient (2.3) below. In section 3, the proofs of Theorems 1.3 and 1.4 are given by using the previous results and the method of moving planes.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Let

$$F_\alpha(u(x)) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy = C_{n,\alpha} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy.$$

Throughout this and next section, we assume

$$(2.1) \quad G \in C^1(R), \quad G(0) = 0, \quad \text{and } G'(t) \geq c_0 > 0, \quad \text{for } t \in R.$$

Using the simple maximum principle in [CLLg], we prove the following strong maximum principle.

Lemma 2.1. Let Ω be a bounded domain in R^n . Assume that $u(x) \in C_{loc}^{1,1}(\Omega) \cap L_\alpha(R^n)$, is lower semi-continuous on $\bar{\Omega}$, and satisfies

$$(2.2) \quad \begin{cases} F_\alpha(u(x)) \geq 0, & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega^c. \end{cases}$$

If $u(x)$ attains 0 somewhere in Σ_λ , then

$$u(x) \equiv 0, \quad x \in R^n.$$

Proof . If $u(x)$ is not identical to 0, there exists an x^0 such that $u(x^0) = 0$ and

$$\begin{aligned} F_\alpha(u(x)) &= \int_{R^n} \frac{G(u(x^0) - u(z))}{|x^0 - z|^{n+\alpha}} dz \\ &= \int_{R^n} \frac{G'(\Psi(z))[u(x^0) - u(z)]}{|x^0 - z|^{n+\alpha}} dz \\ &\leq c_0 \int_{R^n} \frac{-u(z)}{|x^0 - z|^{n+\alpha}} dz \\ &< 0. \end{aligned}$$

This contradicts (2.2) and the proof is ended.

Proof of Theorem 1.1.

If (1.4) does not hold, without loss of generality, we assume $U(x) < 0$ at some point in Ω ; then the lower semi-continuity of $U(x)$ on $\bar{\Omega}$ guarantees that there exists some $\tilde{x} \in \Omega$ such that

$$U(\tilde{x}) = \min_{\Omega} U(x) < 0.$$

And it deduces from the condition (1.3) that \tilde{x} is in the interior of Ω . By the defining integral, we have

$$\begin{aligned} F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) &= C_{n,\alpha} P V \int_{R^n} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}} dy \\ &= C_{n,\alpha} P V \int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}} dy \\ &+ C_{n,\alpha} P V \int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u(y)) - G(u(\tilde{x}) - u_\lambda(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \\ &\leq C_{n,\alpha} P V \int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u_\lambda(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \\ &+ C_{n,\alpha} P V \int_{\Sigma_\lambda} \frac{G(u_\lambda(\tilde{x}) - u(y)) - G(u(\tilde{x}) - u_\lambda(y))}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \\ &= C_{n,\alpha} P V \int_{\Sigma_\lambda} \frac{2G'(\cdot)U(\tilde{x})}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \\ (2.3) \quad &\leq 2C_{n,\alpha} c_0 U(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy. \end{aligned}$$

Let $D = B_{2l}(\tilde{x}) \cap \tilde{\Sigma}_\lambda$, then

$$\begin{aligned}
 \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy &\geq \int_D \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \\
 (2.4) \qquad \qquad \qquad &\geq \frac{1}{10} \int_{B_{2l}(\tilde{x})} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \\
 &\geq \frac{1}{l^\alpha}.
 \end{aligned}$$

Thus from (2.3),

$$(2.5) \qquad F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) \leq \frac{CU(\tilde{x})}{l^\alpha} < 0.$$

Together (2.5) with (1.3), it yields

$$(2.6) \qquad U(\tilde{x}) \geq -Cc_1(\tilde{x})l^\alpha V(\tilde{x}) \text{ and } V(\tilde{x}) \leq 0.$$

We know from (2.6) that there exists \bar{x} such that

$$V(\bar{x}) = \min_{\Omega} V(x) < 0.$$

Similarly to (2.5), it derives that

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) \leq \frac{CV(\bar{x})}{l^\beta} < 0.$$

Combining it with (2.6), we have for l sufficiently small,

$$\begin{aligned}
 0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_2(\bar{x})U(\bar{x}) \\
 &\leq \frac{CV(\bar{x})}{l^\beta} + c_2(\bar{x})U(\tilde{x}) \\
 &\leq C\left(\frac{V(\bar{x})}{l^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V(\tilde{x})\right) \\
 &\leq C\left(\frac{V(\bar{x})}{l^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V(\bar{x})\right) \\
 &\leq C\frac{V(\bar{x})}{l^\beta}(1 - c_1(\tilde{x})c_2(\bar{x})l^{\alpha+\beta}) \\
 &< 0.
 \end{aligned}$$

This contradiction shows that (1.4) must be true.

Next we prove (1.5). Without loss of generality, let us suppose that there exists $\eta \in \Omega$ such that

$$U(\eta) = 0.$$

Then we use $\frac{1}{|x-y|} > \frac{1}{|x-y^\lambda|}$, for $x, y \in \Sigma_\lambda$, to have

$$\begin{aligned}
(2.7) \quad & F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) \\
&= C_{n,\alpha} PV \int_{R^n} \frac{G(u_\lambda(\eta) - u_\lambda(y)) - G(u(\eta) - u(y))}{|\eta - y|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(\eta) - u_\lambda(y)) - G(u(\eta) - u(y))}{|\eta - y|^{n+\alpha}} dy \\
&+ C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(\eta) - u(y)) - G(u(\eta) - u_\lambda(y))}{|\eta - y^\lambda|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_\lambda} [G(u_\lambda(\eta) - u_\lambda(y)) - G(u(\eta) - u(y))] \left(\frac{1}{|\eta - y|^{n+\alpha}} - \frac{1}{|\eta - y^\lambda|^{n+\alpha}} \right) dy \\
&+ C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(\eta) - u(y)) - G(u(\eta) - u_\lambda(y)) + G(u_\lambda(\eta) - u_\lambda(y)) - G(u(\eta) - u(y))}{|\eta - y^\lambda|^{n+\alpha}} dy \\
&= C_{n,\alpha} G'(\cdot) \int_{\Sigma_\lambda} (U(\eta) - U(y)) \left(\frac{1}{|\eta - y|^{n+\alpha}} - \frac{1}{|\eta - y^\lambda|^{n+\alpha}} \right) dy \\
&+ C_{n,\alpha} G'(\cdot) \int_{\Sigma_\lambda} \frac{2U(\eta)}{|\eta - y^\lambda|^{n+\alpha}} dy \\
&\leq -C c_0 \int_{\Sigma_\lambda} U(y) \left(\frac{1}{|\eta - y|^{n+\alpha}} - \frac{1}{|\eta - y^\lambda|^{n+\alpha}} \right) dy.
\end{aligned}$$

If $U(x) \not\equiv 0$, then (2.7) implies

$$F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) < 0.$$

Using it with (1.3), it shows $V(\eta) < 0$. This is a contradiction with (1.4). Hence $U(x)$ must be identically 0 in Σ_λ . Since

$$U(x^\lambda) = -U(x), \quad x \in \Sigma_\lambda,$$

it gives

$$U(x) \equiv 0, \quad x \in R^n.$$

Again from (1.3), we see

$$V(x) \leq 0, \quad x \in \Sigma_\lambda.$$

Since we already know

$$V(x) \geq 0, \quad x \in \Sigma_\lambda,$$

it must hold

$$V(x) = 0, \quad x \in \Sigma_\lambda.$$

Recalling $V(x^\lambda) = -V(x)$, we arrive at

$$V(x) \equiv 0, \quad x \in R^n.$$

Similarly, one can show that if $U(x)$ or $V(x)$ attains 0 at one point in Σ_λ , then both $U(x)$ and $V(x)$ are identically 0 in R^n . This completes the proof.

Proof of Theorem 1.2. There exists $\tilde{x} \in \Omega$, such that

$$U(\tilde{x}) = \min_{\Omega} U(x) < 0.$$

Using (2.3), we have

$$\begin{aligned} F_{\alpha}(u_{\lambda}(\tilde{x})) - F_{\alpha}(u(\tilde{x})) &= C_{n,\alpha} PV \int_{\Sigma_{\lambda}} \frac{G(u_{\lambda}(\tilde{x}) - u_{\lambda}(y)) - G(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}} \\ &\quad + C_{n,\alpha} PV \int_{\Sigma_{\lambda}} \frac{G(u_{\lambda}(\tilde{x}) - u(y)) - G(u(\tilde{x}) - u_{\lambda}(y))}{|\tilde{x} - y|^{n+\alpha}} \\ &\leq C_{n,\alpha} PV \int_{\Sigma_{\lambda}} \frac{G'(\cdot) 2U(\tilde{x})}{|\tilde{x} - y|^{n+\alpha}} dy \\ &\leq 2C_{n,\alpha} c_0 U(\tilde{x}) \int_{\Sigma_{\lambda}} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy. \end{aligned}$$

For each fixed λ , there exists $C > 0$ such that for $\tilde{x} \in \Sigma_{\lambda}$ and $|\tilde{x}|$ sufficiently large,

$$(2.8) \quad \int_{\Sigma_{\lambda}} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \geq \int_{B_{3|\tilde{x}|}(\tilde{x}) \setminus B_{2|\tilde{x}|}(\tilde{x})} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \sim \frac{C}{|\tilde{x}|^{\alpha}}.$$

Hence

$$(2.9) \quad F_{\alpha}(u_{\lambda}(\tilde{x})) - F_{\alpha}(u(\tilde{x})) \leq \frac{CU(\tilde{x})}{|\tilde{x}|^{\alpha}} < 0.$$

Together (2.9) with (1.6), it is easy to deduce

$$(2.10) \quad V(\tilde{x}) < 0,$$

and

$$(2.11) \quad U(\tilde{x}) \geq -C c_1(\tilde{x}) |\tilde{x}|^{\alpha} V(\tilde{x}).$$

From (2.10), there exists \bar{x} such that

$$V(\bar{x}) = \min_{\Omega} V(x) < 0.$$

Similarly to (2.8), we can derive

$$(2.12) \quad F_{\beta}(v_{\lambda}(\bar{x})) - F_{\beta}(v(\bar{x})) \leq \frac{CV(\bar{x})}{|\bar{x}|^{\beta}} < 0.$$

Combing (1.6) and (2.11), we have for λ sufficiently negative,

$$\begin{aligned} 0 &\leq F_{\beta}(v_{\lambda}(\bar{x})) - F_{\beta}(v(\bar{x})) + c_2(\bar{x})U(\bar{x}) \\ &\leq \frac{CV(\bar{x})}{|\bar{x}|^{\beta}} + c_2(\bar{x})U(\bar{x}) \\ &\leq C \left(\frac{V(\bar{x})}{|\bar{x}|^{\beta}} - c_2(\bar{x})c_1(\tilde{x})|\tilde{x}|^{\alpha}V(\tilde{x}) \right) \\ &\leq \frac{CV(\bar{x})}{|\bar{x}|^{\beta}} (1 - c_1(\tilde{x})|\tilde{x}|^{\alpha}c_2(\bar{x})|\bar{x}|^{\beta}). \end{aligned}$$

It follows that $1 \leq c_1(\tilde{x})|\tilde{x}|^\alpha c_2(\bar{x})|\bar{x}|^\beta$. However, from (1.7) we have $c_1(\tilde{x})|\tilde{x}|^\alpha c_2(\bar{x})|\bar{x}|^\beta < 1$ for $|\tilde{x}|$ and $|\bar{x}|$ sufficiently large. This contradiction shows that (1.8) must be true.

3. SYMMETRY OF SOLUTIONS IN THE WHOLE SPACE R^n

Proof of Theorem 1.3. Choose an arbitrary direction as the x_1 -axis. Let $T_\lambda = \{x \in R^n \mid x_1 = \lambda\}$, $x^\lambda = (2\lambda - x_1, x')$, $u_\lambda(x) = u(x^\lambda)$, $\Sigma_\lambda = \{x \in R^n \mid x_1 < \lambda\}$,

$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

Step1. *Start moving the plane T_λ from $-\infty$ to the right in x_1 -direction.*

We will show that for λ sufficiently negative,

$$(3.1) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

For the fixed λ and $x \in \Sigma_\lambda$, by (1.9),

$$u(x) \rightarrow 0, \quad \text{as } |x| \rightarrow +\infty.$$

As $|x| \rightarrow +\infty$, we have $|x^\lambda| \rightarrow +\infty$; it follows that

$$u_\lambda(x) = u(x^\lambda) \rightarrow 0.$$

Thus for $x \in \Sigma_\lambda$,

$$(3.2) \quad U_\lambda(x) \rightarrow 0, \quad \text{as } |x| \rightarrow +\infty.$$

Similarly, one can show that for $x \in \Sigma_\lambda$,

$$V_\lambda(x) \rightarrow 0, \quad \text{as } |x| \rightarrow +\infty.$$

By the mean value theorem it is easy to see that

$$F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) = f(v_\lambda(x)) - f(v(x)) = f'(\xi_\lambda(x))V_\lambda(x),$$

and

$$F_\beta(v_\lambda(x)) - F_\beta(v(x)) = g(u_\lambda(x)) - g(u(x)) = g'(\eta_\lambda(x))U_\lambda(x),$$

where $\xi_\lambda(x)$ is valued between $v_\lambda(x)$ and $v(x)$; $\eta_\lambda(x)$ is valued between $u_\lambda(x)$ and $u(x)$. By Theorem 1.2, it suffices to check the decay rate of $f'(\xi_\lambda(x))$ and $g'(\eta_\lambda(x))$, at the points where $V_\lambda(x)$ and $U_\lambda(x)$ are negative respectively. Since $u_\lambda(x) < u(x)$ and $v_\lambda(x) < v(x)$, we have

$$0 \leq u_\lambda(x) \leq \eta_\lambda(x) \leq u(x), \quad 0 \leq v_\lambda(x) \leq \xi_\lambda(x) \leq v(x).$$

At those points for $|x|$ sufficiently large, the decay assumptions (1.9) and (1.10) instantly yields that

$$c_1(x) = f'(\xi_\lambda(x)) \sim o\left(\frac{1}{|x|^\alpha}\right), \quad c_2(x) = g'(\eta_\lambda(x)) \sim o\left(\frac{1}{|x|^\beta}\right).$$

Consequently, there exists $R_0 > 0$, such that, if \tilde{x} and \bar{x} are negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ in Σ_λ respectively, then by Theorem 1.2, it holds that

$$(3.3) \quad |\tilde{x}| \leq R_0 \quad \text{or} \quad |\bar{x}| \leq R_0.$$

Without loss of generality, we may assume

$$(3.4) \quad |\tilde{x}| \leq R_0.$$

For λ sufficiently negative, combining (3.2) with fact that

$$U_\lambda(x) = 0, \quad x \in T_\lambda,$$

we know if $U_\lambda(x) < 0$ in Σ_λ , then $U_\lambda(x)$ must have a negative minimum in Σ_λ . This contradicts (3.4). Hence, for λ sufficiently negative we have

$$(3.5) \quad U_\lambda(x) \geq 0,$$

it follows that $V_\lambda(x) \geq 0$ in Σ_λ . Otherwise, there exists \bar{x} in Σ_λ such that

$$V_\lambda(\bar{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0,$$

from (2.12), we have

$$(3.6) \quad F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) < 0.$$

However, combining (1.6) with (3.5), we have $F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) \geq 0$. This is a contradiction with (3.6) and $V_\lambda(x)$ cannot attain its negative value in Σ_λ . It follows that (3.1) must be true. This completes the preparation for the moving planes.

Step 2. *Keep moving the planes to the right to the limiting position T_{λ_0} as long as (3.1) holds.*

Let

$$\lambda_0 = \sup\{\lambda \mid U_\mu(x), V_\mu(x) \geq 0, \quad x \in \Sigma_\mu, \quad \mu \leq \lambda\}.$$

Obviously,

$$(3.7) \quad \lambda_0 < \infty.$$

Otherwise, for any $\lambda > 0$,

$$u(0^\lambda) > u(0) > 0, \quad v(0^\lambda) > v(0) > 0.$$

Meanwhile,

$$u(0^\lambda) \sim \frac{1}{|0^\lambda|^\beta}, \quad v(0^\lambda) \sim \frac{1}{|0^\lambda|^\alpha}, \quad \lambda \rightarrow \infty.$$

This is a contradiction and (3.7) is proved.

Now, we point out that

$$(3.8) \quad U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.$$

Otherwise, we will show that the plane T_λ can be moved further to the right. More rigorously, there exists some $\epsilon > 0$, such that for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$ we have

$$(3.9) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

This is a contradiction with the definition of λ_0 . Hence we must have (3.8).

Now we to prove (3.9) by using Theorem 1.1 and Theorem 1.2.

Suppose (3.8) is false, then $U_{\lambda_0}(x) \geq 0$ and $V_{\lambda_0}(x) \geq 0$ are positive somewhere in Σ_{λ_0} , and Theorem 1.1 gives

$$U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}.$$

Let R_0 be determined in Theorem 1.2. It follows that for any $\delta > \epsilon > 0$,

$$U_{\lambda_0}(x) \geq c_0 > 0, \quad V_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}.$$

From the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to λ , there exists $\epsilon > 0$, such that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, we have

$$(3.10) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}.$$

Suppose that (3.9) is false, we have $U_\lambda(x) < 0$, $V_\lambda(x) < 0$, $x \in \Sigma_\lambda$. If \tilde{x} and \bar{x} are negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ in Σ_λ respectively. Next we consider two possibilities.

Case 1. One of the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ lies in $B_{R_0}(0)$, i.e. it is in the narrow region $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$. The other is outside of $B_{R_0}(0)$. Without loss of generality, we may assume the negative minimum of $U_\lambda(x)$ lies in $B_{R_0}(0)$. from (2.6), we have

$$(3.11) \quad U_\lambda(\tilde{x}) \geq -c_1(\tilde{x})l^\alpha V_\lambda(\tilde{x}).$$

Furthermore, we know

$$\begin{aligned} 0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_2(\bar{x})U_\lambda(\bar{x}) \\ &\leq \frac{CV_\lambda(\bar{x})}{|\bar{x}|^\beta} + c_2(\bar{x})U_\lambda(\tilde{x}) \\ &\leq C\left\{\frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V_\lambda(\tilde{x})\right\} \\ &\leq C\left\{\frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V_\lambda(\bar{x})\right\} \\ &\leq C\frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta}[1 - c_1(\tilde{x})l^\alpha c_2(\bar{x})|\bar{x}|^\beta]. \end{aligned}$$

Hence

$$(3.12) \quad 1 \leq c_1(\tilde{x})l^\alpha c_2(\bar{x})|\bar{x}|^\beta.$$

From (1.7), we know that $c_2(\bar{x})|\bar{x}|^\beta$ is small for $|\bar{x}|$ sufficiently large. Since $l = \epsilon + \delta$ is very narrow and $c_1(\tilde{x})$ is bounded from below in $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$, $c_1(\tilde{x})l^\alpha$ can be small. Consequently, $c_1(\tilde{x})l^\alpha c_2(\bar{x})|\bar{x}|^\beta < 1$. This is a contradiction with (3.12) and (3.9) is proved.

Case 2. The negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ lie in $B_{R_0}(0)$, i.e. they are all in the narrow region $\Sigma_{\lambda_0+\epsilon} \setminus \Sigma_{\lambda_0-\delta}$.

By (2.5),

$$(3.13) \quad F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) \leq \frac{CU_\lambda(\tilde{x})}{l^\alpha} < 0,$$

where $l = \delta + \epsilon$. Together with (1.3), it implies

$$(3.14) \quad U_\lambda(\tilde{x}) \geq -c_1(\tilde{x})l^\alpha V_\lambda(\tilde{x}).$$

Similarly to (3.13), we derive

$$F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) \leq \frac{CV_\lambda(\bar{x})}{l^\beta} < 0.$$

Noting (3.14), we have for l sufficiently small,

$$\begin{aligned}
0 &\leq F_\beta(v_\lambda(\bar{x})) - F_\beta(v(\bar{x})) + c_2(\bar{x})U_\lambda(\bar{x}) \\
&\leq \frac{CV_\lambda(\bar{x})}{l^\beta} + c_2(\bar{x})U_\lambda(\tilde{x}) \\
&\leq C\left\{\frac{V_\lambda(\bar{x})}{l^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V_\lambda(\tilde{x})\right\} \\
&\leq C\left\{\frac{V_\lambda(\bar{x})}{l^\beta} - c_2(\bar{x})c_1(\tilde{x})l^\alpha V_\lambda(\bar{x})\right\} \\
&\leq C\frac{V_\lambda(\bar{x})}{l^\beta}[1 - c_1(\tilde{x})c_2(\bar{x})l^{\alpha+\beta}] \\
&< 0.
\end{aligned}$$

This contradiction shows that (3.9) must be true.

Now we have shown that $U_{\lambda_0}(x) \equiv 0$, $V_{\lambda_0}(x) \equiv 0$, $x \in \Sigma_{\lambda_0}$. Since the x_1 direction can be chosen arbitrarily, we actually prove that $u(x)$ and $v(x)$ must be radially symmetric about some point x^0 . Also the monotonicity follows easily from the argument.

This completes the proof of Theorem 1.3.

4. NON-EXISTENCE OF SOLUTIONS ON A HALF SPACE R_+^n

We investigate the system (1.2).

Proof of Theorem 1.4. Based on (1.11) and $f(0) = 0$, $g(0) = 0$, one can see from the proof of Lemma 2.1 that

$$\text{either } u(x) > 0, v(x) > 0 \text{ or } u(x) \equiv 0, v(x) \equiv 0, \text{ for } x \in R_+^n.$$

In fact, without loss of generality, assume $u(x) \not\equiv 0$, there exists x^0 such that $u(x^0) = 0$, and

$$F_\alpha(u(x^0)) = c_{n,\alpha}PV \int_{R^n} \frac{G(u(x^0) - u(y))}{|x^0 - y|^{n+\alpha}} dy < 0,$$

i.e. $0 \leq f(v(x)) = F_\alpha(u(x)) < 0$, this is impossible. Hence if $u(x)$ or $v(x)$ attains 0 somewhere in R_+^n , then $u(x) = v(x) \equiv 0$, $x \in R_+^n$.

Hence in the following, we assume that $u(x) > 0$ and $v(x) > 0$ in R_+^n . Let us carry on the method of moving planes on the solution u along x_n direction.

Define $T_\lambda = \{x \in R^n | x_n = \lambda\}$, $\lambda > 0$, $\Sigma_\lambda = \{x \in R^n | 0 < x_n < \lambda\}$. Let $x^\lambda = (x_1, \dots, x_{n-1}, 2\lambda - x_n)$ be the reflection of x about the plane T_λ , and $U_\lambda(x) = u_\lambda(x) - u(x)$, $V_\lambda(x) = v_\lambda(x) - v(x)$.

The key ingredient (2.3) is obtained in this proof of Theorem 1.1. To see that it still applies in this situation, we only need to take $\Sigma = \Sigma_\lambda \cup R_-^n$, where $R_-^n = \{x \in R^n | x_n \leq 0\}$.

Step1. For λ sufficiently small, we have immediately

$$(4.1) \quad U_\lambda(x) \geq 0, V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda,$$

since Σ_λ is a narrow region.

Step2. Since (4.1) provides a starting point, we move the plane T_λ upward as long as (4.1) holds. Define

$$\lambda_0 = \sup\{\lambda > 0 | U_\mu(x) \geq 0, V_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda\}.$$

We show that

$$(4.2) \quad \lambda_0 = \infty.$$

Otherwise, if $\lambda_0 < \infty$, then using (4.1), Theorem 1.1, Theorem 1.2 and going through the similar arguments as in Section 3, we are able to show

$$U_{\lambda_0} \equiv 0, V_{\lambda_0} \equiv 0, x \in \Sigma_{\lambda_0},$$

which implies

$$\begin{aligned} u(x_1, \dots, x_{n-1}, 2\lambda_0) &= u(x_1, \dots, x_{n-1}, 0) = 0, \\ v(x_1, \dots, x_{n-1}, 2\lambda_0) &= v(x_1, \dots, x_{n-1}, 0) = 0. \end{aligned}$$

This is impossible, because we assume that $u(x), v(x) > 0$ in R_+^n .

Therefore, (4.2) must be valid and the solutions $u(x), v(x)$ are increasing with respect to x_n . This contradicts (1.11).

This completes the proof of Theorem 1.4.

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